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Closed-vortex-type solitons with Hopf index

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Abstract. The structure of three-dimensional solitons with non-trivial Hopf index is investigated for the S^2 nonlinear σ-model. It is shown that the corresponding regular solutions are of closed-vortex type. We prove the existence of regular vortex-like solutions which are used for the approximation of solitons with large values of Hopf index.

1. Introduction

It has been shown (Rybakov 1979) that three-dimensional scalar charged localised solutions to nonlinear field equations (stationary solitons) may only be conditionally stable, that is stable under the condition of charge fixation. But for solitons carrying topological charges, this condition holds automatically. Hence, topological solitons are the most promising candidates for the role of stable extended particles.

The least number of independent field variables required to introduce a non-trivial topological charge in (3 + 1)-dimensional space–time is equal to two. The corresponding topological charge Q is the Hopf index of the mapping $S^3 \to S^2$. In this case the field N_a, $a = 1, 2, 3$, takes values in a two-sphere, i.e. $N_a(t, \boldsymbol{x}): R \times R^3 \to S^2$ and $N_a^2 = 1$. We suppose the natural boundary condition

$$N_a(t, \infty) = \delta_{a3}. \tag{1}$$

In the present paper we study a generalisation of the Faddeev model (Vakulenko and Kapitanski 1979, Vladimirov 1980) which is the simplest model with Hopf index. The construction of the Faddeev model from an N_a field is based on the following ideas. To obtain finite-energy soliton solutions in a three-dimensional space one should include in the Lagrangian terms containing space derivatives in more than second power to overcome the obstacles of Derrick's theorem (Derrick 1964). The time derivatives, on the other hand, should enter the model only in quadratic form.

A more detailed study of this model will be made in § 3.

2. Structure of the Hopf index

If we introduce the vector A_μ by

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = 2\epsilon^{abc}\partial_\mu N_a \partial_\nu N_b N_c, \tag{2}$$

then the degree of knottedness of the vector-field lines $\boldsymbol{B} = \text{curl}\,\boldsymbol{A}$ will be the Hopf index Q which may be given by (Nicole 1978, de Vega 1978)

$$Q = -(8\pi)^{-2} \int \boldsymbol{A} \cdot \boldsymbol{B} \, d^3x = \int J^0 \, d^3x \tag{3}$$

where the identically conserved topological current is

$$J^\mu = -(128\pi^2)^{-1} \epsilon^{\mu\nu\sigma\tau} F_{\nu\sigma} A_\tau.$$

We investigate the case of the N-field invariant under the group

$$G = \text{diag}[O(2)_I \otimes O(2)_S], \tag{4}$$

$O(2)_I$ and $O(2)_S$ being the groups of rotations around the axes N_3 and z respectively. It is immediately apparent that the G-invariant N-field should satisfy the condition

$$(T_3)_S N_a + [(T_3)_I N]_a = 0 \tag{5}$$

where $(T_3)_S$ and $(T_3)_I$ are the generators of the groups $O(2)_S$ and $O(2)_I$, respectively. If we now expand the invariance group, the field N_a must also satisfy the additional equation

$$[(T_3)_I N]_a + d(T_j)_S N_a = 0, \qquad d = \text{constant} \neq 0, \, j \neq 3,$$

where $(T_j)_S$ are the generators of $O(3)_S$. Then from (5) one obtains

$$[(T_3)_S - d(T_j)_S] N_a = 0$$

which shows that the vector ∇N_a has poloidal structure. Therefore, since the vector field $\boldsymbol{B}$ may be expressed as $\boldsymbol{B} = (2/N_3)[\nabla N_1 \times \nabla N_2]$, it is clear that the corresponding $\boldsymbol{B}$-lines will be circles with their centres lying on a certain straight line. Consequently, the degree of knottedness of the $\boldsymbol{B}$-lines or Q is trivial. Thus, we have established that any attempt at expanding the invariance group (4) gives only trivial values of Q for invariant fields. Hence, the group (4) is the maximal compact group, invariant fields of which satisfy the condition $Q \neq 0$. In this case the Hopf index may be calculated explicitly.

As it is seen from (5), the G-invariant field in spherical coordinates (r, ϑ, α) should be of the form

$$N_3 = \cos\beta = w(r, \vartheta), \qquad \gamma = \tan^{-1}(N_2/N_1) = m\alpha + v(r, \vartheta), \qquad m \in \mathbb{Z}, \tag{6}$$

with w and v being arbitrary functions of r, ϑ and (β, γ) polar angles of the N_a-field. From (3), using the Biot–Savart law, we find that

$$Q = -2(8\pi)^{-3} \iint d^3x \, d^3x' [\boldsymbol{B}' \cdot (\boldsymbol{R} \times \boldsymbol{B})]/R^3 \tag{7}$$

where

$$\boldsymbol{R} = \boldsymbol{x} - \boldsymbol{x}', \qquad R = |\boldsymbol{R}|, \qquad \boldsymbol{B} = -2(\nabla w \times \nabla \gamma), \qquad \boldsymbol{B}' = \boldsymbol{B}(\boldsymbol{x}').$$

The ansatz (6) gives

$$\boldsymbol{B} = -(2m/r^2 \sin\vartheta)(\boldsymbol{e}_r w_\vartheta - \boldsymbol{e}_\vartheta r w_r) - 2\boldsymbol{e}_\alpha K/r \tag{8}$$

where $K = w_r v_\vartheta - w_\vartheta v_r$. From the structure of (8) and finiteness of $\boldsymbol{B}$ it follows that $\nabla w = 0$ on the z axis. Hence we find $w \to 1$ for $r \sin\vartheta \to 0$, which shows that the surface

w = constant is homeomorphic to tore T^2. Inserting (8) in (7), through some mathematical manipulations we finally arrive at the expression

$$Q=\frac{m}{4\pi}\int_0^\infty \mathrm{d}r\int_0^\pi \mathrm{d}\vartheta\,(1-w)(w_r v_\vartheta - w_\vartheta v_r). \tag{9}$$

In cylindrical coordinates (ρ, z, α) for $w(\rho, z)=w(\rho, -z)$ and $Q\neq 0$ we have $v(\rho, z)=-v(\rho, -z)$. It is clear from (6) that v is not a single-valued function and since $v\in[-\pi n/2, \pi n/2]$, $n\in\mathbb{Z}$, it may have jumps $[v]=\epsilon n\pi$, $\epsilon=\pm 1$. Let us denote by $C(\rho, z)$ the jump-line of the function $v(\rho, z)$. Now, noticing that $2(w-1)(w_\rho v_z - w_z v_\rho)$ is the α component of the vector curl $[v\nabla(1-w)^2]$ and applying Stokes' theorem to (9) relative to the contour $\Gamma_+\cup\Gamma_-$ (figure 1), we conclude that due to the boundary

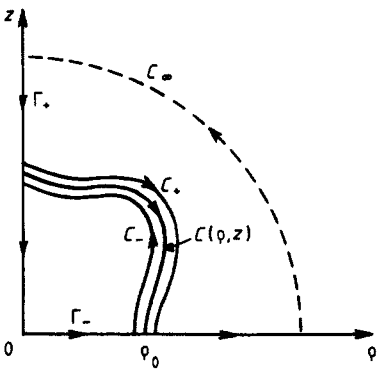

Figure 1. $\Gamma_\pm$ are closed contours of integration for calculating the Hopf index using (9). $C_\pm$ are parts of the contours adjoining the jump-line $C(\rho, z)$. The broken line C_∞ is the arc of a circle of infinite radius.

conditions on $w(\rho, z)$ only the paths C_- and C_+ adjoining the jump-line $C(\rho, z)$ (see figure 1) contribute to the integral. Hence, taking the above facts into account, one obtains

$$Q=\frac{m}{4\pi}\int_{C_+\cup C_-} v(\mathrm{d}\boldsymbol{l}\cdot\nabla)(1-w)^2=\frac{m}{4\pi}[v]\int_{C(\rho,z\geqslant 0)}(\mathrm{d}\boldsymbol{l}\cdot\nabla)(1-w)^2. \tag{10}$$

Note that due to the single-valuedness of the N-field the endpoint $C(\rho_0, 0)$, where $v=0$, $\pm\pi n/2$, corresponds to the north or south pole of S^2. Since the first possibility implies $Q=0$, it follows that $w=-1$ at $C(\rho_0, 0)$. Hence, supposing the uniqueness of the endpoint $C(\rho_0, 0)$, we obtain finally

$$Q=\epsilon nm=\pm nm. \tag{11}$$

Thus, identification of the jump-line of v helps one to calculate the Hopf index easily from (10) or (11).

3. S^2 nonlinear σ-model

As is well known (Vladimirov 1980), under the condition $N_a^2=1$ only two independent Poincaré and $O(3)_I$ invariants, $(\partial_\mu N_a)^2$ and $F^2_{\mu\nu}$, can be constructed. The Lagrangian density of the model may therefore be given by

$$\mathscr{L} = -(\varepsilon^2/4)F_{\mu\nu}^2 + \lambda^2(\partial_\mu N_a)^2 - M^2(1-w) \tag{12}$$

where ε, λ, M are constant parameters and $F_{\mu\nu}$ is defined by (2). The massive term in the generalised Faddeev model (12) that guarantees the exponential vanishing of the field solution at space infinities breaks the symmetry $O(3)_I$ to $O(2)_I$, which is however already broken by the boundary condition (1)†.

Vakulenko and Kapitanski (1979) have shown that the Hamiltonian H of model (12) allows an estimate from below through the Hopf index Q in the form

$$H > C_0|Q|^{3/4} \tag{13}$$

but the authors did not provide the value of C_0. Taking into account the minimum value of the constant C in the Sobolev inequality $\|\phi^3\| \leqslant C\|\nabla\phi\|^3$ (Rosen 1971), it can be shown that

$$C_0 = \varepsilon\lambda(4\pi)^2\sqrt{2}3^{3/8}.$$

Using the ansatz (6) for $m = 1$, the static Hamiltonian reduces to

$$H = \int d^3x \, \{2\varepsilon^2 \sin^2\beta[(\nabla\beta)^2/\rho^2 + (\nabla\beta \times \nabla v)^2] + \lambda^2(\nabla\beta)^2$$
$$+ \lambda^2 \sin^2\beta[\rho^{-2} + (\nabla v)^2] + 2M^2\sin^2(\beta/2)\}. \tag{14}$$

For investigating the structure of fields minimising the functional (14), we recall that the surface β = constant is homeomorphic to T^2. Note that for small values of $Q = 1, 2, \ldots$ the field solution is complicated, but when $Q = n \gg 1$, since the torus is converted into a closed vortex of small curvature (as will be shown later), the field structure is much simplified in this case. Hence we investigate first the infinite vortex (Enz 1977) for which

$$\beta = \beta(\rho), \qquad v = kz, \qquad k = \text{constant}.$$

Denoting $\rho\lambda/\varepsilon\sqrt{2} = s$, $2k^2\varepsilon^2/\lambda^2 = c^2$ and $2M^2\varepsilon^2/\lambda^4 = \nu$, we find that the energy of the vortex of length l is

$$H = 2\pi l\lambda^2 I[\beta] \tag{15}$$

where

$$I[\beta] = \int_0^\infty \frac{ds}{s}\{(\beta')^2[s^2 + \sin^2\beta(1 + c^2s^2)] + \sin^2\beta(1 + c^2s^2) + 2\nu s^2\sin^2(\beta/2)\}. \tag{16}$$

The existence of vortex solutions may be proved by a variational method, if we assume boundary conditions

$$\beta(0) = \pi, \qquad \beta(\infty) = 0. \tag{17}$$

Using (17) and inequalitites like $(\beta')^2s^2 + \sin^2\beta \geqslant 2s|\beta'\sin\beta|$, after simple calculations one obtains the estimate

$$I[\beta] \geqslant (\pi^2c^2 + 16)^{1/2} + (\pi^2c^2 + 128\nu/9)^{1/2}$$

which guarantees the existence of a minimising sequence $\{\beta_n(s)\}$ for the functional (16).

† Some additional investigations show that the proposed massive term is the simplest one, for which the Hamiltonian estimate from below through Q exists even when gauge vector fields are included (Kundu 1981).

On the other hand, from the finiteness of the integral $\int_0^\infty (\mathrm{d}s/s) y_n'^2 \leqslant 2C_1^2 < \infty$ where $y_n = \cos \beta_n$, and Schwartz's inequality, we have

$$y_n(s) - y_n(0) \leqslant \left| \int_0^s y_n' \, \mathrm{d}s \right| \leqslant \left(\int_0^s y_n'^2 \frac{\mathrm{d}s}{s} \right)^{1/2} \left(\int_0^s s \, \mathrm{d}s \right)^{1/2} \leqslant C_1 s. \tag{18}$$

From (18) for $s \leqslant \delta \ll 1$ one obtains

$$|\pi - \beta_n(s)| \leqslant C_2 s^{1/2}. \tag{19}$$

Similarly as in (18), using $\int_0^\infty (\beta_n')^2 s \, \mathrm{d}s \leqslant C_3^2 < \infty$, for $s \geqslant \delta$ we obtain

$$|\beta_n(s) - \beta_n(\delta)| \leqslant C_3[\lg(s/\delta)]^{1/2}. \tag{20}$$

From (19) and (20) it follows that $\beta_n \in H_\beta^1$, where H_β^1 is a Hilbert space with norm

$$\|\beta_n\| = \left(\int_0^\infty (\beta_n')^2 s \, \mathrm{d}s + \beta_n^2(a) \right)^{1/2}, \qquad a \in (0, \infty).$$

Therefore, the sequence $\{\beta_n\}$ tends weakly to some limit function $\beta(s) \in H_\beta^1$. Now, to show that the functional $I[\beta]$ attains its lower bound, we express $I_n = I[\beta_n]$ as the sum of positive scalar products of the elements from $L_2(0, \infty)$:

$$I_n = (g^{(n)}, h^{(n)}) + \sum_{i=1}^{2} \|\rho_i^{(n)}\|^2$$

where

$$g^{(n)} = (\beta_n')^2(1+s^2), \qquad h^{(n)} = [s^2 + \sin^2 \beta_n(1 + c^2 s^2)]/(s + s^3).$$
$$\sqrt{s}\rho_1^{(n)} = \sin \beta_n (1 + c^2 s^2)^{1/2}, \qquad \rho_2^{(n)} = (2\nu s)^{1/2} \sin(\beta_n/2).$$

We are now in a position to use the method developed by Kundu *et al* (1979) to prove that the lower bound of the functional (16) is reachable and also the regularity of the function $\beta(s)$ having the asymptotes

$$\beta \approx \begin{cases} \pi - as & \text{for } s \ll 1, \\ bs^{-1/2} \exp[-s(c^2 + \nu/2)^{1/2}] & \text{for } s \gg 1, \end{cases}$$

a, b being positive constants. The Euler–Lagrange equation obtained from (16) has also been solved numerically for $\nu = 1$ and $c \in (0, 5)$. One such solution is represented graphically in figure 2.

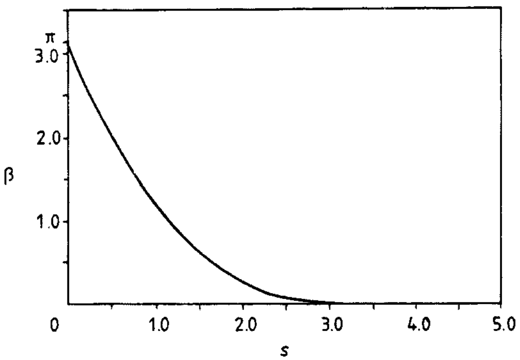

Figure 2. Dependence of β on s. The solution is obtained for $\beta'(0) = -2.4894$ and $c = c_0 = 1.908$.

The vortex solution thus obtained may now be used for an approximation of the soliton solution with large value of Hopf index $Q = n \gg 1$. With this aim we identify the endpoints of the vortex of length l to obtain a closed vortex. Note that the vector lines of the $\boldsymbol{B}$-field are helixes with pitch $2\pi/k$. Hence, for the closed vortex the degree of knottedness of the $\boldsymbol{B}$-lines or the Hopf index is $Q = kl/2\pi = n$, from which $l = 2\pi n/k = \pi 2^{3/2}\varepsilon n/\lambda c$. This geometrical consideration may also be checked directly by substituting $w = w(\rho)$, $v = kz$, $z \in [-l/2, l/2]$ and $m = 1$ in (9) rewritten in cylindrical coordinates:

$$\begin{aligned} Q &= \frac{1}{4\pi}\int_0^\infty d\rho \int_{-l/2}^{+l/2} dz\, (1-w)(w_\rho v_z - w_z v_\rho) \\ &= \frac{kl}{4\pi}\int_0^\infty d\rho\, (1-w)w_\rho \\ &= kl/2\pi \end{aligned} \qquad (21)$$

where we have used boundary conditions (17) for $w = \cos\beta$. It is clear from (21) that for fixed values of $k = c\lambda/\varepsilon\sqrt{2}$ the Hopf index $Q \sim l$. Since our approximation is valid for large values of l, we conclude that the closed-vortex solution found by us should have large Hopf index. Now to find the value of c we minimise the energy of the closed vortex given by

$$H(c) = \varepsilon\lambda n 4\pi^2\sqrt{2}I(c, \nu)/c.$$

Numerical calculations show that the function $I(c, 1)/c$ attains its minimum value (see figure 3) at $c = c_0 \approx 1.908$, where it equals 10.646, which agrees well with the estimate (13) even for $Q = 1$.

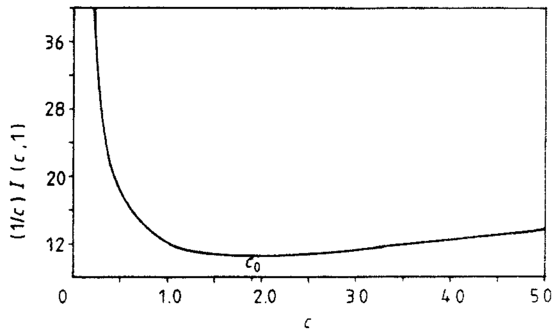

Figure 3. Dependence of the closed-vortex energy on the parameter c.

4. Conclusion

Thus we have investigated the structure of the Hopf index for N-fields in S^2, invariant under the group diag$[O(2)_I \otimes O(2)_S]$, and have established that the soliton solutions with non-trivial Hopf index have the closed-vortex-like structure. We have shown the existence of regular vortex-type solutions in the S^2 nonlinear σ-model and used them for the approximation of soliton solutions with large values of Hopf index. It has also been demonstrated that such a representation agrees well with the known energy estimate for the given model.

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